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STEADY-STATE TEMPERATURE DISTRIBUTION IN AN INHOMOGENEOUS MEDIUM WITH LOCAL INCLUSIONS

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We present a modification of the method of image regions [G. I. Marchuk, Methods of Numerical Mathematics, Springer-Verlag [1975)] to solve the boundary-value problem for the steady-state temperature distribution in an irregular multiply connected region.

We consider the boundary-value problem for the temperature distribution u(x) in the multiply connected region $G=\Pi\setminus\bigcup_{s=1}^{N}\omega_s$. (Fig. 1), where $\Pi=\{(x_1,\ x_2)\colon 0\le x_1\le L,\ 0\le x_2\le l\}$, and ω_s is a region which corresponds to a local inclusion. At the boundary of the inclusion, the heat flux is zero:

$$\operatorname{div}[H(x)\operatorname{grad}u(x)]=-f(x),\ x=(x_1,\ x_2)\in G,$$
$$u|_\Gamma=0,\ \left.\frac{\partial u}{\partial n}\right|_{\gamma_s}=0\ (s=1,\ 2,\ \ldots,\ N). \quad (1)$$

Here $H(x)>0$ is the heat-conduction coefficient of the inhomogeneous medium; $f(x)>0$, volume density of the heat sources; Γ, boundary of the rectangular region Π; γ_s, boundary of the local inclusion ω_s; and n, normal to the contour γ_s.

We shall present a method which makes it possible to find a rigorous solution of problem (1) for any shape and number of local inclusions ω_s. Together with (1) we shall formulate an auxiliary problem in the rectangular region Π:

$$\sum_{m=1}^{2}\frac{\partial}{\partial x_m}\left[\eta(x;\ \varepsilon)\ \frac{\partial v_\varepsilon}{\partial x_m}\right]=-F(x),\ x\in\Pi, \quad (2)$$

$$v_\varepsilon|_\Gamma=0, \quad (3)$$

where $\eta(x;\ \varepsilon)$ and F(x) are piecewise-smooth functions which are defined as follows:

$$\eta(x;\ \varepsilon)=\begin{cases}H(x),\ x\in G,\\ \varepsilon=\text{const}\geqslant 0,\ x\in\Pi\setminus G,\end{cases}\quad F(x)=\begin{cases}f(x),\ x\in G,\\ 0,\ x\in\Pi\setminus G.\end{cases}$$

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Since (2) is an equation of the divergence type, and has piecewise-smooth coefficients, the Dirichlet problem (2), (3) has a unique continuous solution [2, 3] which satisfies the Hölder conditions with respect to the variables x_m. At the lines of discontinuity of function $\eta(x;\varepsilon)$, the derivatives $\partial v_\varepsilon/\partial x_m$ have a first order discontinuity so that the following connection formulas hold:

$$\left(H \frac{\partial v_\varepsilon}{\partial n}\bigg|_{\gamma_s}\right)^+ = \left(\varepsilon \frac{\partial v_\varepsilon}{\partial n}\bigg|_{\gamma_s}\right)^-,$$

where the subscripts + and − denote the corresponding values of the functions at the different sides of the contours γ_s. It is clear that for $\varepsilon = 0$ these conditions take the form

$$\frac{\partial v_0}{\partial n}\bigg|_{\gamma_s} = 0.$$

Therefore, the solution of the original boundary-value problem (1) is obtained by restricting the solution $v_0(x)$ of the problem (2), (3) which is continuous in the region Π to the region G for $\varepsilon = 0$.

The auxiliary problem will be solved by reduction to an infinite system of linear algebraic equations [4]. Let us consider in the rectangle Π the following orthogonal systems of functions:

$$X_{kn}(x) = \frac{2}{\sqrt{Ll}} \sin\frac{k\pi x_1}{L} \sin\frac{n\pi x_2}{l},$$

$$Y_{kn}^{(1)}(x) = \frac{2}{\sqrt{Ll}} \cos\frac{k\pi x_1}{L} \sin\frac{n\pi x_2}{l},$$

$$Y_{kn}^{(2)}(x) = \frac{2}{\sqrt{Ll}} \sin\frac{k\pi x_1}{L} \cos\frac{n\pi x_2}{l},$$

$$Z_{kn}(x) = \frac{2}{\sqrt{Ll}} \cos\frac{k\pi x_1}{L} \cos\frac{n\pi x_2}{l}.$$

Noting that the function $v_\varepsilon(x)$ vanishes at the contour Γ, we shall seek the solution of Eq. (2) which belongs to the space $C_{0,\alpha}(\bar{\Pi}) \cap \overset{\circ}{W}{}_2^1(\Pi)$ [3] in the form of the following double trigonometric series:

$$v_\varepsilon(x) = \sum_{k,n=1}^{\infty} a_{kn}(\varepsilon) X_{kn}(x). \tag{4}$$

Since the function $v_\varepsilon(x)$ belongs in the region Π to the Hölder class $C_{0,\alpha}$ it follows [5] that the series on the right-hand side of Eq. (4) tends to $v_\varepsilon(x)$ uniformly. In addition, noting that the function $v_\varepsilon(x)$ is positive which follows from the physical formulation of the problem with $f(x) > 0$, and using the generalized mean value theorem, we obtain

$$a_{kn} = \frac{2}{\sqrt{Ll}} \int_0^L \sin\frac{k\pi x_1}{L}\, dx_1 \int_0^l v_\varepsilon(x_1, x_2) \sin\frac{n\pi x_2}{l}\, dx_2 =$$

$$= \frac{2}{\sqrt{Ll}} \sin\frac{n\pi \bar{x}_2}{l} \int_0^L w(x_1) \sin\frac{k\pi x_1}{L}\, dx_1 \quad (0 < \bar{x}_2 < l).$$

Since the function

$$w(x_1) = \int_0^l v_\varepsilon(x_1, x_2)\, dx_2$$

is continuous in the interval [0, L], and vanishes at its ends, the following relation holds:

$$a_{kn} = O\left(\frac{1}{k^2}\right), \quad k \to \infty.$$

Analogously, we obtain

$$a_{kn} = O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

Consequently, we have the following estimate for the coefficients a_{kn} of expansion (4):

$$|a_{kn}| \leqslant \frac{C}{k^2 + n^2}, \quad C = \text{const} > 0.$$

We shall now determine the values of these coefficients. Substituting the assumed form of solution (4) into Eq. (2), we obtain

$$\sum_{m=1}^{2} \frac{\partial}{\partial x_m} \left[\eta(x;\ \varepsilon) \sum_{k,n=1}^{\infty} \alpha_{kn}^{(m)} a_{kn}(\varepsilon)\, Y_{kn}^{(m)}(x)\right] = -F(x), \tag{5}$$

where

$$\alpha_{kn}^{(m)} = \begin{cases} k\pi/L, & \text{if} \quad m = 1, \\ n\pi/l, & \text{if} \quad m = 2. \end{cases}$$

We now define the function

$$\Phi^{(m)}(x;\ \varepsilon) = \eta(x;\ \varepsilon) \sum_{i,j=1}^{\infty} \alpha_{ij}^{(m)} a_{ij}(\varepsilon)\, Y_{ij}^{(m)}(x) \tag{6}$$

and, noting that the functions $F(x)$ and $\Phi^{(m)}(x;\ \varepsilon)$ belong to the space $L_2(\Pi)$, we can expand these functions in a double trigonometric series in the rectangle Π:

$$F(x) = \sum_{k,n=1}^{\infty} q_{kn} X_{kn}(x), \tag{7}$$

$$\Phi^{(m)}(x;\ \varepsilon) = \sum_{k,n=0}^{\infty} \lambda_{kn}^{(m)} \varphi_{kn}^{(m)}(\varepsilon)\, Y_{kn}^{(m)}(x). \tag{8}$$

Here,

$$\lambda_{kn}^{(m)} = \begin{cases} \lambda_k, & \text{if} \quad m = 1, \\ \lambda_n, & \text{if} \quad m = 2, \end{cases} \quad \lambda_k = \begin{cases} 1/2, & \text{if} \quad k = 0, \\ 1, & \text{if} \quad k \neq 0. \end{cases}$$

Using the expansions (7) and (8), we obtain from Eq. (5)

$$\sum_{m=1}^{2} \alpha_{kn}^{(m)} \varphi_{kn}^{(m)}(\varepsilon) = q_{kn} \quad (k,\ n = 1,\ 2,\ \ldots). \tag{9}$$

We note that the expansion coefficients of an arbitrary function $g(x) \in L_2(\Pi)$ in a double trigonometric Fourier series of orthogonal functions $X_{kn}(x)$, $Y_{kn}^{(m)}(x)$, $Z_{kn}(x)$ are, respectively, the scalar products $(g,\ X_{kn})$, $(g,\ Y_{kn}^{(m)})$, $(g,\ Z_{kn})$ in the space $L_2(\Pi)$.

Using (6), we find for the Fourier coefficients $\varphi_{kn}^{(m)}(\varepsilon)$

$$\varphi_{kn}^{(m)}(\varepsilon) = (\Phi^{(m)}(x;\ \varepsilon),\ Y_{kn}^{(m)}(x)) = \iint_{(\Pi)} \Phi^{(m)}(x;\ \varepsilon)\, Y_{kn}^{(m)}(x)\, dx =$$

$$= \frac{1}{2\sqrt{Ll}} \sum_{i,j=1}^{\infty} \alpha_{ij}^{(m)} a_{ij}(\varepsilon) \{(\eta,\ Z_{k-i,\, n-j}) + (-1)^m (\eta,\ Z_{k-i,\, n+j}) + (-1)^{m-1} (\eta,\ Z_{k+i,\, n-j}) + (\eta,\ Z_{k+i,\, n+j})\}. \tag{10}$$

Denoting the Fourier coefficients of the functions $\eta(x;\ \varepsilon) \in L_2(\Pi)$ in the expansion in a double trigonometric series in terms of the functions $Z_{kn}(x)$ which are orthogonal in Π by $\sigma_{kn}(\varepsilon)$ and noting that $\sigma_{kn}(\varepsilon) = (\eta,\ Z_{kn})$, expression (10) for $\varphi_{kn}^{(m)}(\varepsilon)$ can be written in the form

$$\varphi_{kn}^{(m)}(\varepsilon) = \frac{1}{2\sqrt{Ll}} \sum_{i,j=1}^{\infty} \alpha_{ij}^{(m)} a_{ij}(\varepsilon) \{\sigma_{k-i,\, n-j}(\varepsilon) + (-1)^m \sigma_{k-i,\, n+j}(\varepsilon) + (-1)^{m-1} \sigma_{k+i,\, n-j}(\varepsilon) + \sigma_{k+i,\, n+j}(\varepsilon)\}. \tag{11}$$

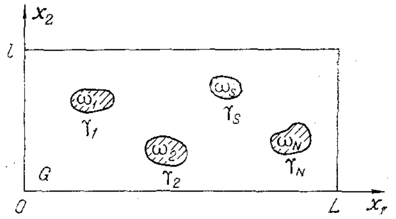

Fig. 1

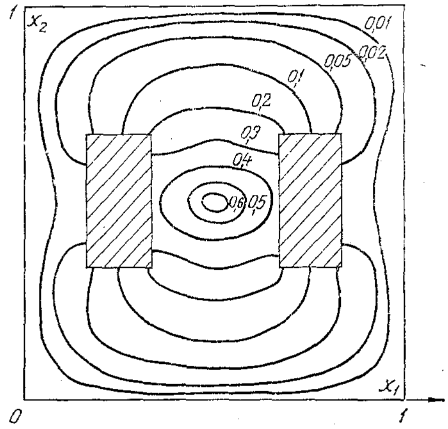

Fig. 2

Fig. 1. Multiply connected region with local inclusions.

Fig. 2. Temperature field in a multiply connected region with local inclusions.

Using (11), we can now write relations (9) in the form

$$\sum_{i,j=1}^{\infty}{}' D_{knij} a_{ij}(\varepsilon) = q_{kn} \ (k,\ n = 1,\ 2,\ \ldots), \tag{12}$$

where

$$D_{knij} = \frac{\pi^2}{2\sqrt{Ll}} \left\{ \left(\frac{ki}{L^2} + \frac{nj}{l^2} \right) [\sigma_{k-i,\, n-j}(\varepsilon) - \sigma_{k+i,\, n+j}(\varepsilon)] - \left(\frac{ki}{L^2} - \frac{nj}{l^2} \right) [\sigma_{k-i,\, n+j}(\varepsilon) - \sigma_{k+i,\, n-j}(\varepsilon)] \right\}.$$

We shall number consecutively the elements of the infinite matrix (a_{kn}) in such a way that each pair of indices k and n corresponds to a number μ according to the rule

$$\mu = \frac{1}{2}(k+n)(k+n-1) - n - 1. \tag{13}$$

Analogously,

$$\nu = \frac{1}{2}(i+j)(i+j-1) - j - 1. \tag{14}$$

According to the rules (13) and (14), we put

$$\xi_\mu(\varepsilon) = a_{kn}(\varepsilon),\ \tau_\mu = q_{kn},\ S_{\mu\nu} = D_{knij}$$

and write relations (12) in the form

$$\sum_{\nu=1}^{\infty} S_{\mu\nu}\xi_\nu(\varepsilon) = \tau_\mu \ (\mu = 1,\ 2,\ \ldots). \tag{15}$$

We note that the matrix elements $S_{\mu\nu}$ at the principal diagonal exceed in absolute value the remaining matrix elements. Equation (15) can therefore be transformed into the following system of algebraic equations of the second kind:

$$\xi_\mu(\varepsilon) + \sum_{\substack{\nu=1 \\ \nu\neq\mu}} Q_{\mu\nu}\xi_\nu(\varepsilon) = r_\mu \ (\mu = 1,\ 2,\ \ldots),$$
$$Q_{\mu\nu} = \frac{S_{\mu\nu}}{W_\mu},\ \Gamma_\mu = \frac{\tau_\mu}{W_\mu},\ W_\mu = S_{\mu\mu}. \tag{16}$$

Clearly, the correspondence (k, n) $\to \mu$ defined by (13) is one to one in the region of whole numbers, and for each $\mu = 1, 2, \ldots$ one can find the indices k and n from the formula

$$k=\mu-\frac{1}{2}E(\mu)[E(\mu)-1],\ n=\frac{1}{2}E(\mu)[E(\mu)+1]-\mu+1, \tag{17}$$

where $E(\mu)$ is the whole part of the real number $1/2+\sqrt{2\mu}$. The following estimates hold:

$$k=O(\mu^{1/2}),\ n=O(\mu^{1/2}),\ n\to\infty.$$

Then, using the estimate

$$|\sigma_{kn}(\varepsilon)|\leqslant\frac{C}{k+n},\ C=\text{const},$$

for the Fourier coefficients of piecewise-smooth function $\eta(x;\varepsilon)$ which is nonnegative in the region Π, one can show analogously to [6, 7] that the reduction method [8] is applicable for the approximate solution of the infinite system (16). Having determined the number sequence $\{\xi_\mu(\varepsilon)\}_{\mu=1}^{\infty}$ from the infinite system (16), the two-dimensional sequence $\{a_{kn}\}_{k,n=1}^{\infty}$ can then be constructed using the correspondences (17).

It should be noted that the coefficients $Q_{\mu\nu}$ of the unknowns $\xi_\nu(\varepsilon)$ in the equations of the infinite system (16) depend on the parameter ε which enters explicitly only in values of the Fourier coefficients $\sigma_{kn}(\varepsilon)$ of the function $\eta(x;\varepsilon)$, and

$$\sigma_{kn}(\varepsilon)=\iint\limits_{(G)}H(x)Z_{kn}(x)\,dx+\varepsilon\iint\limits_{(\Pi\setminus G)}Z_{kn}(x)\,dx.$$

Therefore, if we put $\varepsilon=0$ in the solution of the infinite system (16), then the function

$$v_0(x)=\sum_{k,n=1}^{\infty}a_{kn}(0)X_{kn}(x)$$

gives a rigorous solution of the original boundary-value problem (1) in the multiply connected region G.

Figure 2 shows the example of a numerical calculation of the temperature field induced by a local point heat source when $f(x)=A\delta(x_1-L/2,x_2-l/2)$, and for the following parameters of the problem: $L=1$, $l=1$, $H=1$, $A=36$. The rectangular local inclusions are shaded. The steady-state temperature distribution is shown in the form of isothermal lines.

In conclusion, we note that the numerical calculation using the above scheme uses only the standard procedures for the calculation of Fourier coefficients and the solution of a linear algebraic system of equations. The suggested method for the solution of the boundary value problem (1) is therefore sufficiently effective and simple for engineering calculations.

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